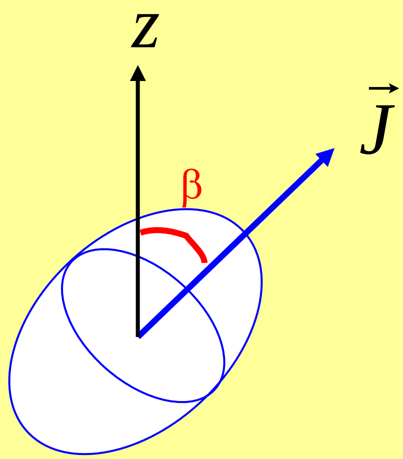
Assume that the charge distribution is an ellipsoid of revolution with symmetry axis along the total angular momentum vector J:



As for the magnetic dipole moment, we specify the **intrinsic** electric quadrupole moment as the expectation value when J is maximally aligned with z:

$$Q_{\text{int}} = \left\langle \hat{E}_2 \right\rangle \Big|_{m_J = J}$$

"Quantum geometry":

$$\cos\beta = \frac{m_J}{|J|} = \frac{J}{\sqrt{J(J+1)}}$$

If an electric field gradient is applied along the z-axis as shown, the observable energy will shift by an amount corresponding to the intrinsic quadrupole moment transformed to a coordinate system rotated through angle β to align with the z-axis

Result:

Let Q_{lab} be the electric quadrupole moment we **measure** for the ellipsoidal charge distribution with $m_J = J$:

$$Q_{\text{lab}} = \underbrace{\frac{1}{2}\left(3\cos^2\beta - 1\right)}_{\substack{\text{standard, classical} \\ \text{prescription for} \\ \text{rotated coordinates}}} Q_{\text{int}} = \underbrace{\left(\frac{J-1/2}{J+1}\right)}_{\substack{\text{“Quantum geometry”} \\ \text{for the rotation function}}} Q_{\text{int}}$$

<u>How do we apply this to anything?</u>

1. A spherically symmetric state (L = 0) has $Q_{int} = 0$ (e.g. deuteron S-state)

2. Even a distorted state with J = ½ will not have an observable quadrupole moment

3. J = 1, L = 2 is the smallest value of total angular momentum for which we can observe a nonzero quadrupole moment

→ **these are the quantum numbers for the deuteron D-state!**

recall our model for the deuteron wave function:

$$\left|\psi_d\right\rangle = a\left|{}^3S_1\right\rangle + b\left|{}^3D_1\right\rangle$$

result for the quadrupole moment:

note cancellation here

$$Q_{\text{int}} = \frac{\sqrt{2}}{10}\,|a^* b|\,\left\langle r^2\right\rangle_{SD} - \frac{1}{20}\,b^2\left\langle r^2\right\rangle_{DD}$$

$$= +0.00286 \pm 0.00003 \ \text{bn}$$

A good model of the N-N interaction can fit both the magnetic moment and the quadrupole moment of the deuteron with the same values of a and b! →

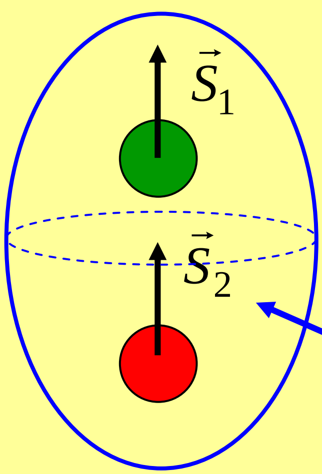
1. Independent of the value of L, the state with intrinsic spins coupled to S = 1 has lower energy

→ this implies a term proportional to:

S=1 has lower energy

$$-\left\langle \vec{S}_1 \bullet \vec{S}_2 \right\rangle = -\tfrac{1}{2}\left\langle S^2 - S_1^2 - S_2^2 \right\rangle = \begin{cases} -1/4, & S=1 \\ +3/4, & S=0 \end{cases}$$

2. The deuteron quadrupole moment implies a non-central component, i.e. the potential is not spherically symmetric. Since the symmetry axis for Q is along J, Q > 0 means that the matter distribution is stretched out along the J – axis:



Q > 0, observed

→ This implies a "tensor" force, proportional to:

$$-\left\langle S_{12} \right\rangle = -\left\langle 3\,\frac{\left(\vec{S}_1 \bullet \vec{r}\right)\left(\vec{S}_2 \bullet \vec{r}\right)}{r^2} - \vec{S}_1 \bullet \vec{S}_2 \right\rangle$$

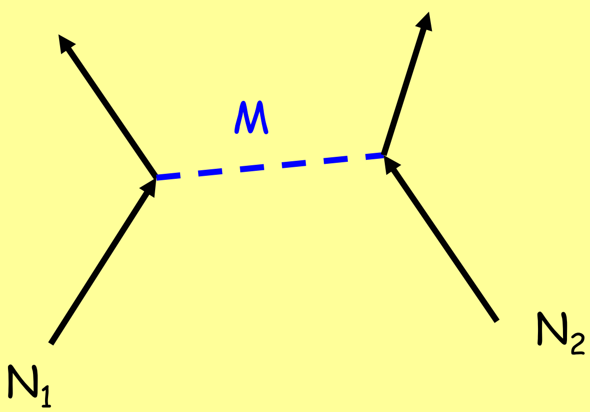
compare: magnetic dipole-dipole interaction, see Griffiths problem 6.20

3. There is also a spin-orbit term, as deduced from N-N scattering experiments with a polarized beam:

$$V_{S-O} \sim \left\langle \vec{L} \bullet \vec{S} \right\rangle$$

(This plays a very important role also in determining the correct order of energy levels in nuclear spectra – more later!)

4. Finally, all contributions to the N-N interaction are based on a microscopic meson exchange mechanism:



Where M is a π, ρ, ω ... meson, etc.

and each term has a spatial dependence of the form:

$$V(r) = g \, \frac{e^{-m\,r}}{r} \times (spin\ \ function)$$

State of the art N-N interaction model: *(not to scare anybody...)*

PHYSICS REPORTS (Review Section of Physics Letters) 149, No. 1 (1987) 1–89. North-Holland, Amsterdam

(89 page exposition of one of only ~3 state-of-the-art models of the N-N interaction worldwide – constantly refined and updated since first release.)

THE BONN MESON-EXCHANGE MODEL FOR THE NUCLEON–NUCLEON INTERACTION*

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Brace yourself – it takes 2 pages just to write all the terms down!!!

pseudoscalar mesons:

(tensor operator)

$$V_{\rm ps}(m_{\rm ps}, r) = \frac{1}{12}\frac{g_{\rm ps}^2}{4\pi} m_{\rm ps}\left[\left(\frac{m_{\rm ps}}{m}\right)^2 Y(m_{\rm ps}r)\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 + Z(m_{\rm ps}r)\,S_{12}\right]; \tag{F.6}$$

(they use "σ" for spin)

scalar mesons:

$$V_{\rm s}(m_{\rm s}, r) = -\frac{g_{\rm s}^2}{4\pi} m_{\rm s}\left\{\left[1-\frac{1}{4}\left(\frac{m_{\rm s}}{m}\right)^2\right]Y(m_{\rm s}r) + \frac{1}{4m^2}\left[\nabla^2 Y(m_{\rm s}r) + Y(m_{\rm s}r)\nabla^2\right] + \tfrac{1}{2}Z_1(m_{\rm s}r)\boldsymbol{L}\cdot\boldsymbol{S}\right\}; \tag{F.7}$$

(spin-orbit interaction)

vector mesons:

$$\begin{aligned} V_{\rm v}(m_{\rm v}, r) = \frac{g_{\rm v}^2}{4\pi} m_{\rm v}&\left\{\left[1+\frac{1}{2}\left(\frac{m_{\rm v}}{m}\right)^2\right]Y(m_{\rm v}r) - \frac{3}{4m^2}\left[\nabla^2 Y(m_{\rm v}r) + Y(m_{\rm v}r)\nabla^2\right]\right. \\ &\left.+\frac{1}{6}\left(\frac{m_{\rm v}}{m}\right)^2 Y(m_{\rm v}r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \tfrac{3}{2}Z_1(m_{\rm v}r)\boldsymbol{L}\cdot\boldsymbol{S} - \tfrac{1}{12}Z(m_{\rm v}r)S_{12}\right\} \\ &+\frac{1}{2}\frac{g_{\rm v}f_{\rm v}}{4\pi} m_{\rm v}\{(m_{\rm v}/m)^2\; Y(m_{\rm v}r) + \tfrac{2}{3}(m_{\rm v}/m)^2\; Y(m_{\rm v}r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 \\ &-4Z_1(m_{\rm v}r)\boldsymbol{L}\cdot\boldsymbol{S} - \tfrac{1}{3}Z(m_{\rm v}r)S_{12}\} \\ &+\frac{f_{\rm v}^2}{4\pi} m_{\rm v}\{\tfrac{1}{6}(m_{\rm v}/m)^2\; Y(m_{\rm v}r)\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2 - \tfrac{1}{12}Z(m_{\rm v}r)S_{12}\}, \end{aligned}$$

with

"Yukawa functions" and derivatives...

$$Y(x) = \mathrm{e}^{-x}/x \,, \qquad Z(x) = (m_\alpha/m)^2(1 + 3/x + 3/x^2)\,Y(x) \,,$$

$$Z_1(x) = \left(\frac{m_\alpha}{m}\right)^2 (1/x + 1/x^2)\,Y(x) \,, \qquad S_{12} = 3\,\frac{(\boldsymbol{\sigma}_1 \cdot \boldsymbol{r})(\boldsymbol{\sigma}_2 \cdot \boldsymbol{r})}{r^2} - \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2 \,, \tag{F.8}$$

and

$$\nabla^2 = +\frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}\,r - \frac{\boldsymbol{L}^2}{r^2} \,.$$

We use units such that $\hbar = c = 1$ ($\hbar c = 197.3286$ MeV fm). The use of the form factor, eq. (3.3), at each vertex (with $n_\alpha = 1$) leads to the following extended expressions:

$$V_\alpha(r) = V_\alpha(m_\alpha, r) - \frac{\Lambda_{\alpha,2}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2}\,V_\alpha(\Lambda_{\alpha,1}, r) + \frac{\Lambda_{\alpha,1}^2 - m_\alpha^2}{\Lambda_{\alpha,2}^2 - \Lambda_{\alpha,1}^2}\,V_\alpha(\Lambda_{\alpha,2}, r) \,, \tag{F.9}$$

where $\Lambda_{\alpha,1} = \Lambda_\alpha + \varepsilon$, $\Lambda_{\alpha,2} = \Lambda_\alpha - \varepsilon$, $\varepsilon/\Lambda_\alpha \ll 1$. $\varepsilon = 10$ MeV is an appropriate choice.

The full NN potential is the sum of the contributions from six mesons:

$$V(r) = \sum_{\alpha=\pi,\rho,\eta,\omega,\delta,\sigma} V_\alpha(r)$$

Parameters and Predictions:

R. Machleidt et al., The Bonn meson-exchange model for the nucleon–nucleon interaction

Table 14

Meson and low-energy parameters (LEP) for the configuration space one-boson-exchange potential (OBEPR)

deuteron properties:

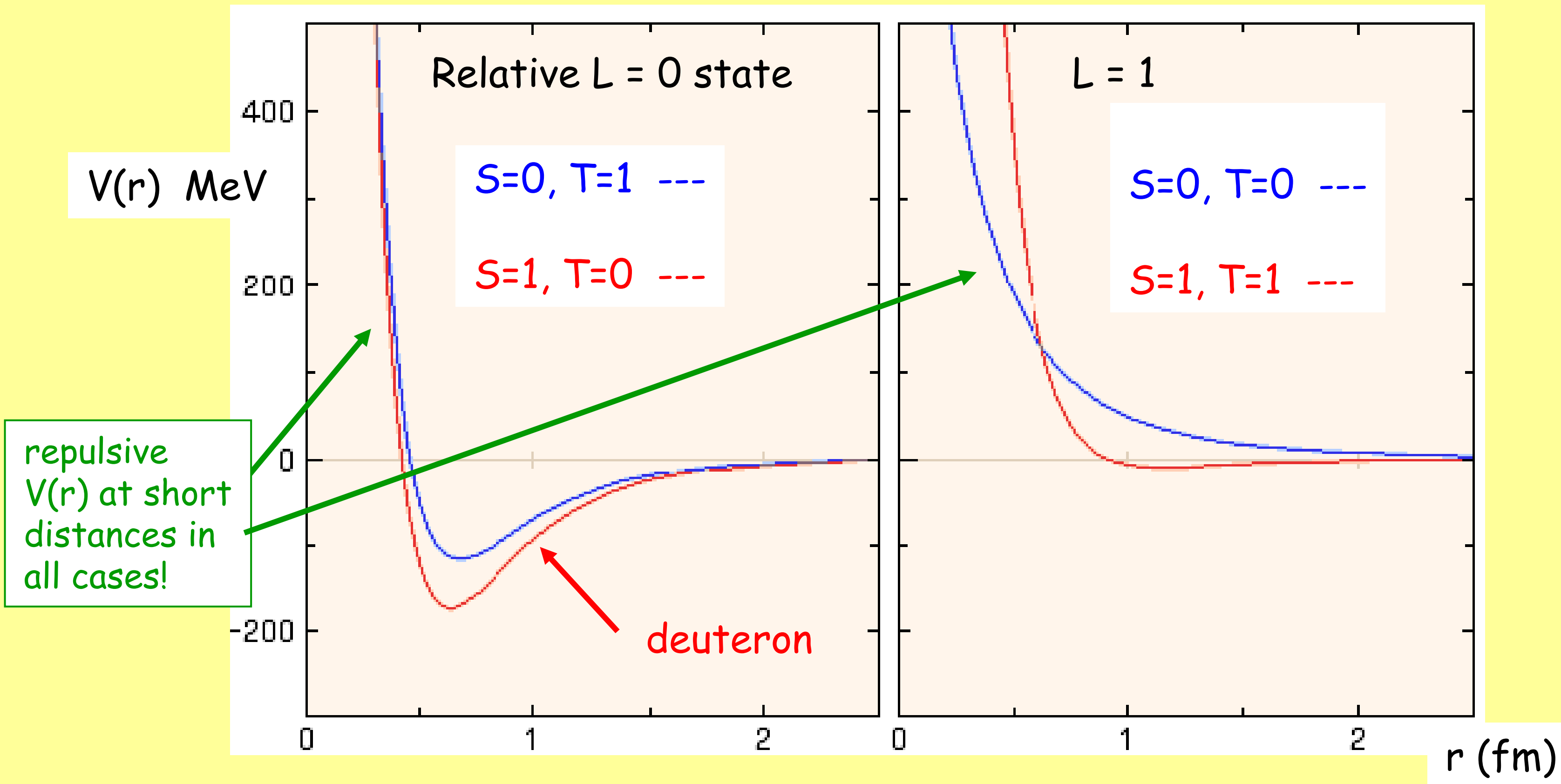
	$g_\alpha^2/4\pi$; $[f_\alpha/g_\alpha]$	m_α (MeV)	Λ_α (GeV)	LEP	Theory
π	14.9	138.03	1.3	ε_d (MeV)	2.2246
				P_D (%)	4.81
ρ	0.95; [6.1]	769	1.3	Q_d (fm^2)	0.274
				μ_d (μ_N)	0.8524
η	3	548.8	1.5	A_S (fm$^{-1/2}$)	0.8860
				D/S	0.0260
ω	20; [0.0]	782.6	1.5	r_d (fm)	1.9691
				a_s (fm)	−23.751
δ	2.6713	983	2.0	r_s (fm)	2.662
				a_t (fm)	5.423
σ	7.7823 [a]	550 [a]	2.0	r_t (fm)	1.759

Impressively good agreement for ~ 10,000 experimental data points in assorted n-p and p-p scattering experiments plus deuteron observables: χ^2/d.f <~ 1.07 !

low energy scattering parameters, etc:

What does the NN potential look like?

It looks different in different spectroscopic states of the 2N system!

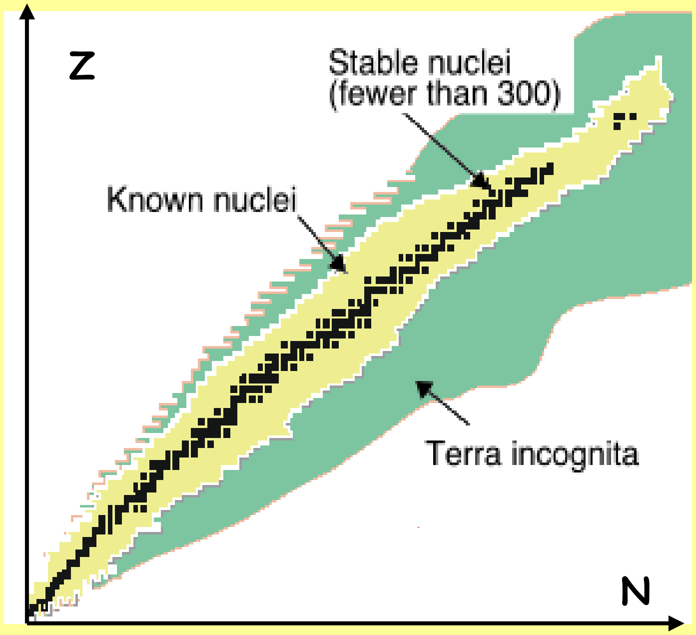


Only the deuteron is bound! Its quantum numbers have the deepest potential well.

Review:

1. Nuclear isotope chart: *(lecture 1)*

- 304 isotopes with $t_{\frac{1}{2}} > 10^9$ yrs *(age of the earth)*
- 177 have even-Z, even-N and $J^\pi = 0^+$
- 121 are even-odd and **only 6 are odd-odd**
- N ≈ Z for light nuclei and N > Z for heavy nuclei



2. Elastic scattering of electrons: *(lecture 7)*

- nuclei are approximately spherical
- RMS charge radius $R = 1.2 \ \mathrm{A}^{1/3} \ \mathrm{fm}$ fitted to electron scattering data
- mass ~ A, and radius ~ $A^{1/3}$ so the density M/V ≈ constant for nuclei (≈ 2×10^{17} kg/m^3), implying that nuclear matter is like an incompressible fluid

Recall from lecture 7 -- Nuclear charge distributions from experiment:

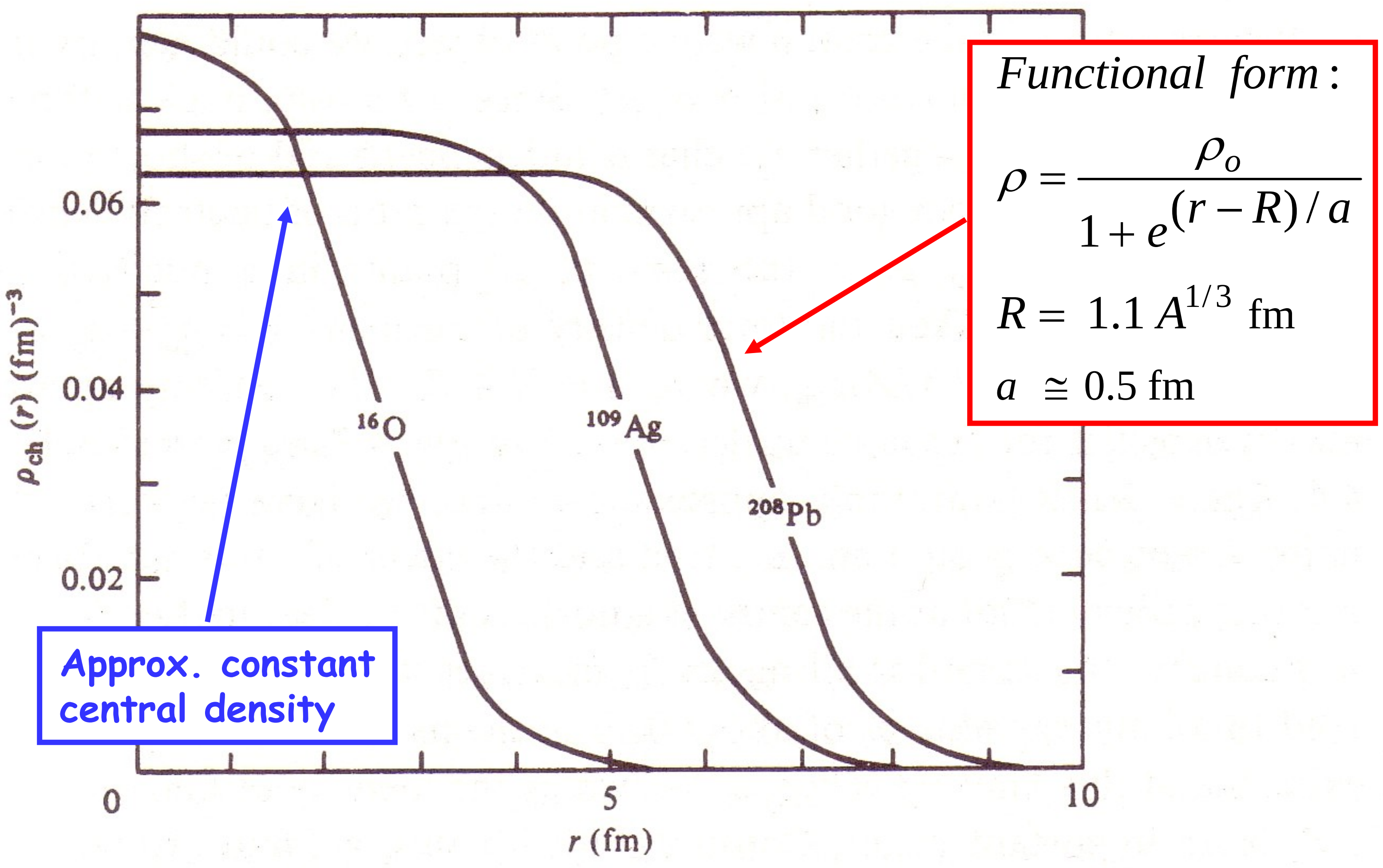
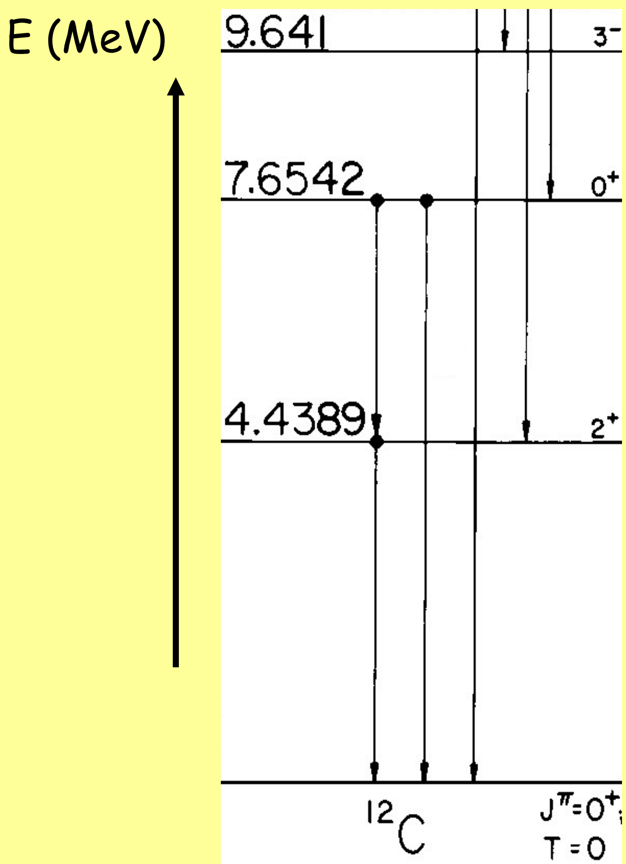
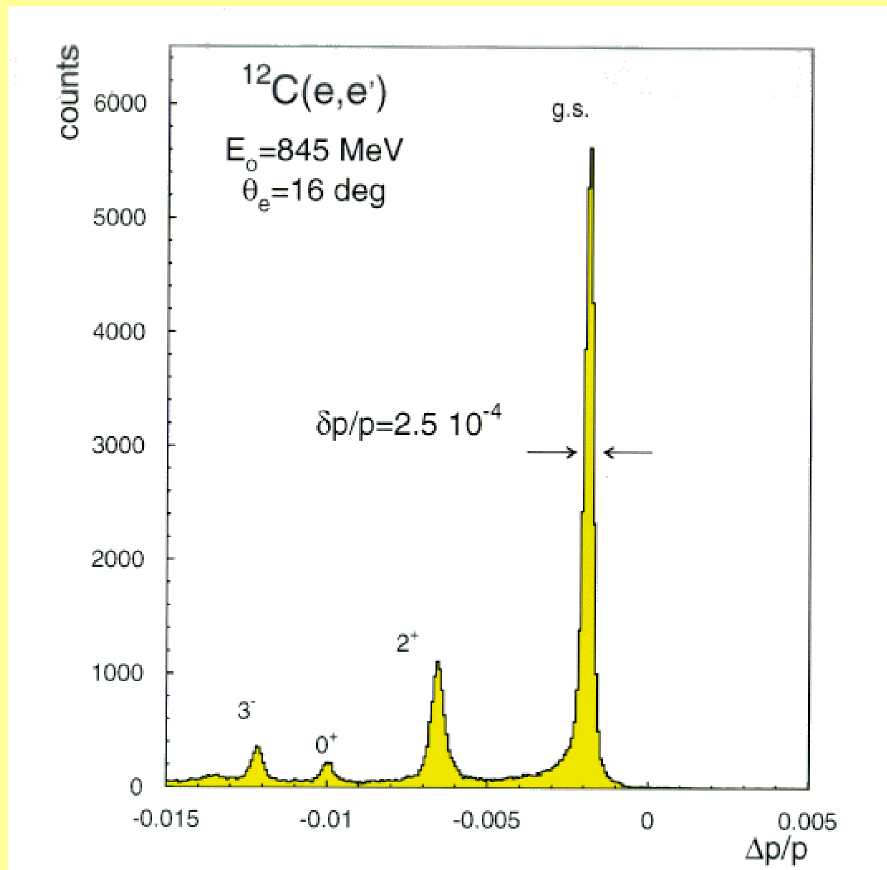


Fig. 4.3 The electric charge density of three nuclei as fitted by $\rho_{ch}(r) = \rho^0_{ch}/[1 + \exp((r - R)/a)]$. The parameters are taken from the compilation in Barrett, R. C. & Jackson, D. F. (1977), *Nuclear Sizes and Structure*, Oxford: Clarendon Press.

3. Inelastic electron scattering: *(lecture 9)*

- Excited states can be identified, on a scale of a few MeV above the ground state, e.g.

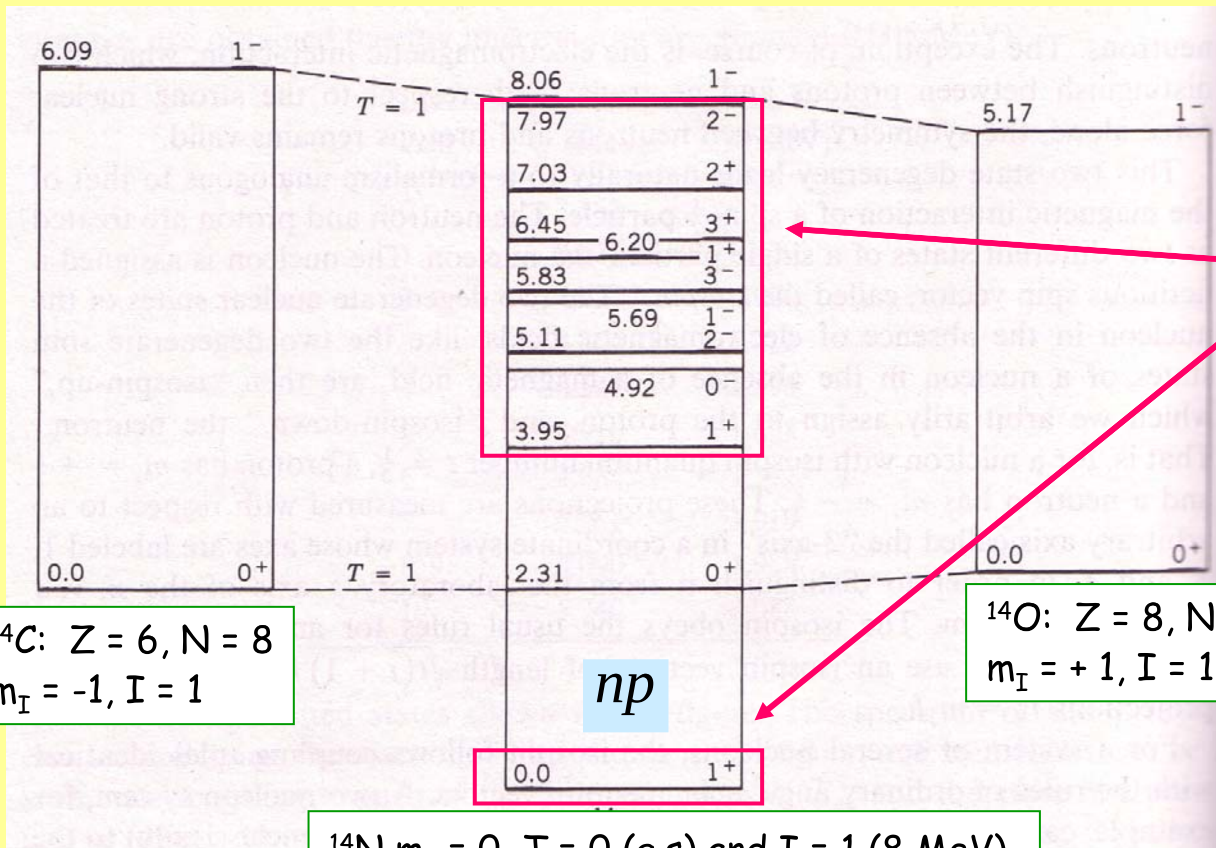




4. Quantum numbers for nuclear states:

- total angular momentum J, parity π
- isospin, I: *(lecture 13)*

  for a **nucleus**, $m_I = \frac{1}{2}(Z-N)$ and $I = |m_I|$, ie lowest energy has smallest I
- Example: "isobaric triplet" ^{14}C, ^{14}N, ^{14}O:



Extra I=0 states at higher energy in the m_I=0 nucleus.

nn

^{14}C: Z = 6, N = 8
m_I = -1, I = 1

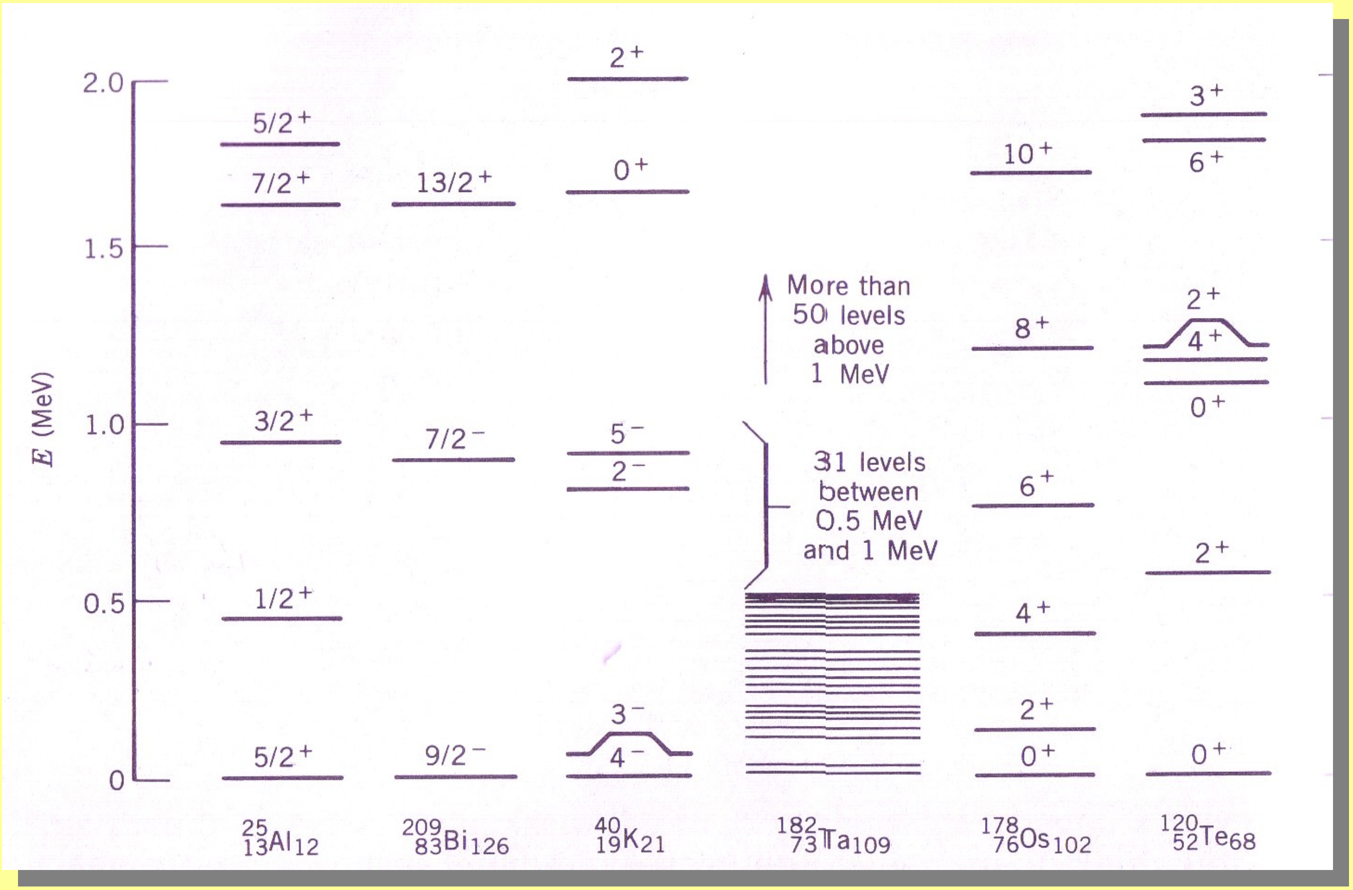
np

^{14}O: Z = 8, N= 6
m_I = + 1, I = 1

pp

^{14}N m_I = 0, T = 0 (g.s) and I = 1 (8 MeV)

Nuclear states in general:

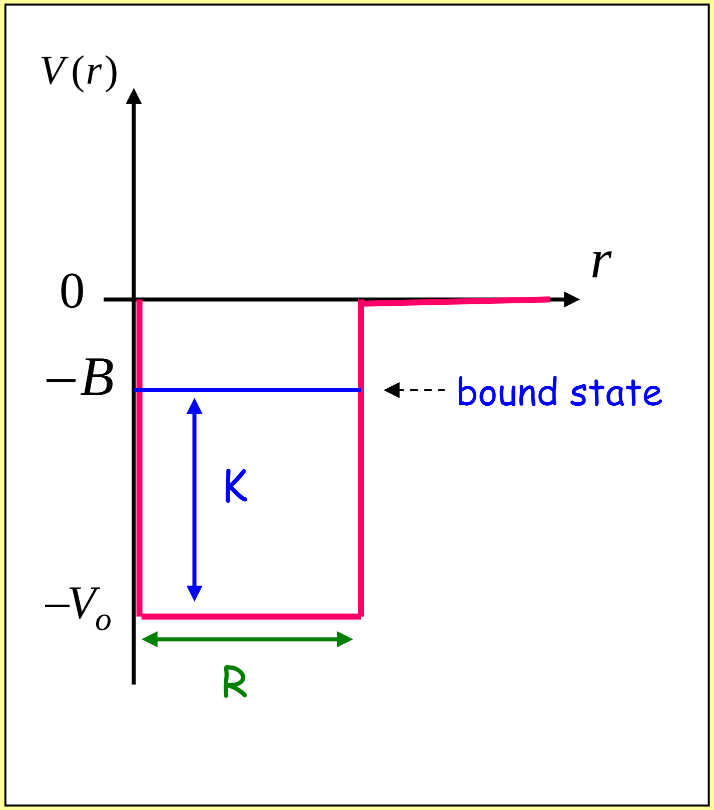


- states have integer or half-integer J depending on whether A is even or odd
- different systematics and energy level spacings for different nuclei
- some nuclei exhibit "single particle" and others "collective" excitations
  → different models to describe this complementary behavior

What is the potential energy function V(r) that nuclei are eigenstates of?

This is not an easy question! The N-N interaction is too complicated to solve in a many-body system: state-of-the-art can go up to $A = 3$!

First approximation: a square well potential, width approx. equal to nuclear radius R:



Assume somehow that we can treat the binding of neutrons and protons like electrons in atoms – individual nucleons have wave functions that are eigenstates of some average nuclear potential V(r).

Each nucleon has a **binding energy B** as shown ($E = -B$)

Kinetic energy $K = (V_0 - B)$

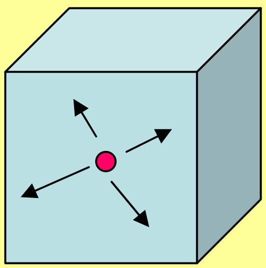
$R \approx 1.2\ A^{1/3}$ fm; most of the wave function is contained inside the well, so this should be approximately the right nuclear size...

Kinetic energy of bound nucleons:

Key point: once we specify the width of the well, the nucleons are confined, and so their kinetic energy is essentially determined by the uncertainty principle:

Simple estimate:

Confining box of side 2 fm. $\Delta p_x \, \Delta x \sim \hbar$



$$\bar{p}_x = 0$$

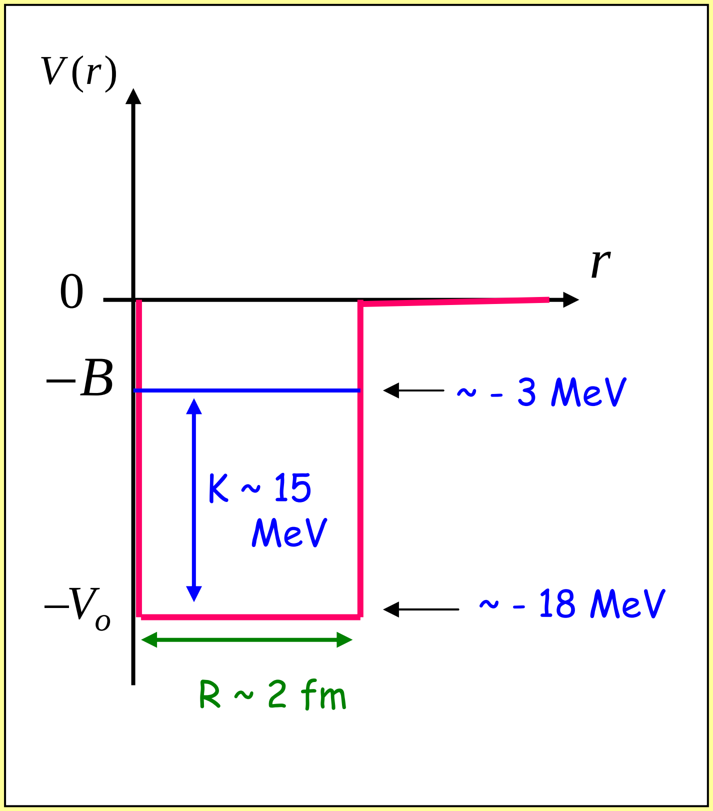
$$\Delta p_x = \sqrt{\left\langle (p_x - \bar{p}_x)^2 \right\rangle} = \sqrt{\left\langle p_x^2 \right\rangle} = \hbar / \Delta x$$

$$\Rightarrow \left\langle p^2 \right\rangle = 3\,(\hbar / \Delta x)^2 \sim 3 \times 10^4 \ \ \mathrm{MeV}^2$$

$$\frac{K}{M} = \frac{\left\langle p^2 \right\rangle}{2M^2} \approx 0.015$$

Conclusion: the motion is non relativistic; K $\approx$ 15 MeV

Now we can specify the potential parameters:



What next?

- We have a complicated system of *A* nucleons.

- About half of them are protons, so a repulsive (+ve energy) term has to be added to the square well to account for this (~ few MeV)

How to connect this model to something observable?

<u>Independent particle model:</u>

- Assume independent particle motion in some average nuclear potential V(r) as shown.

- Then we can fill the eigenstates of the potential to maximum occupancy to form a nucleus, as is done with electrons in atoms (to 1st order…)

- The binding energy of each nucleon, in our model, is a few MeV.
- The potential energy of a bound nucleon is **negative**, by ~ 0.3% of its rest mass energy, which therefore has to show up as a decrease in its mass.
- For A nucleons, the **total binding energy** is:

$$B = \sum_{i=1}^{A} B_i \ = \sum_{i=1}^{A} m_i - \underbrace{M}_{\text{mass of nucleus, } M}$$

The average **binding energy per nucleon, B/A**, can be determined from mass data and used to refine a model for V(r); it ranges systematically from about 1 – 9 MeV as a function of mass number for the stable isotopes.

Reference: F&H ch. 16

- By convention, we set the mass of the carbon-12 **atom** as a standard.
- Denote atomic masses with a "script" **M**, measured in **atomic mass units**, ***U***

M (^{12}C) $\equiv$ 12.0000000000 ..... **U** *(exact!)* → 1 **U** = 931.494 MeV *(expt.)*

Calculation for carbon-12:

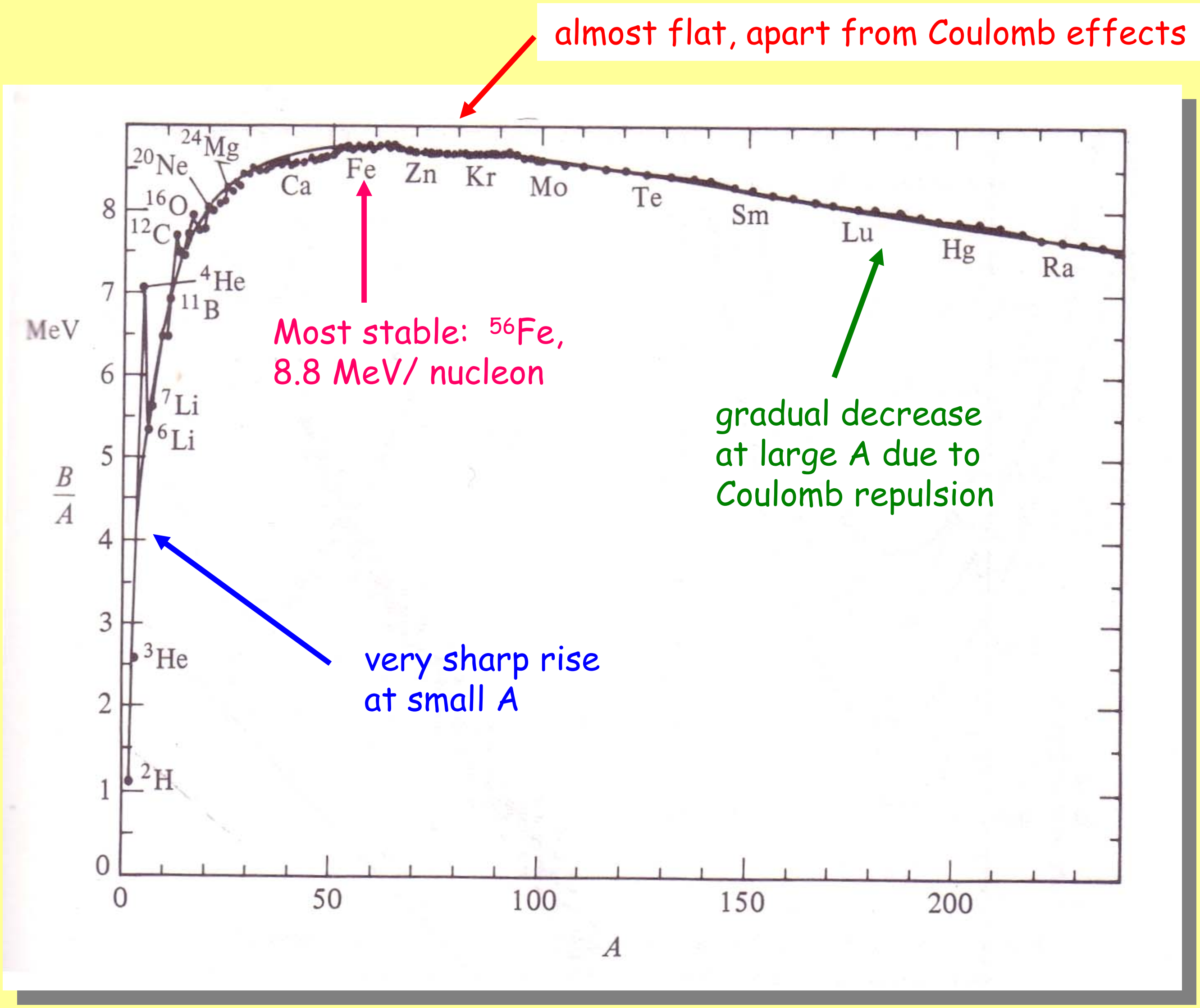
$$\left.\begin{aligned} m_p &= 938.2 \text{ MeV} \\ m_n &= 939.6 \text{ MeV} \\ m_e &= 0.511 \text{ MeV} \end{aligned}\right\} \qquad \begin{aligned} 6 \times \sum_i m_i &= 11{,}269.8 \text{ MeV} \\ 12\,U &= 11{,}178.0 \text{ MeV} \end{aligned}$$

$$\boxed{B\,({}^{12}_{6}C) = \sum_i m_i - M = 91.8 \text{ MeV}}$$

Binding energy per nucleon in ^{12}C: B/A = 7.8 MeV;

Contrast to the deuteron ^{2}H: B/A = 1.1 MeV

The famous Binding Energy per Nucleon curve:



Greater binding energy implies lower mass, greater stability.

Energy is released when configurations of nucleons change to populate the larger B/A region → nuclear energy generation, e.g.

Fusion reactions at small *A* release substantial energy **because the B/A curve rises faster than a straight line at small *A*:**

$$^{2}\mathrm{H} + {}^{2}\mathrm{H} \Rightarrow {}^{3}\mathrm{He} + \mathrm{n} + 3.27\ \mathrm{MeV}$$

$$\Rightarrow {}^{3}\mathrm{H} + \mathrm{p} + 4.03\ \mathrm{MeV}$$

$$^{2}\mathrm{H} + {}^{3}\mathrm{H} \Rightarrow {}^{4}\mathrm{He} + \mathrm{n} + 17.6\ \mathrm{MeV}$$

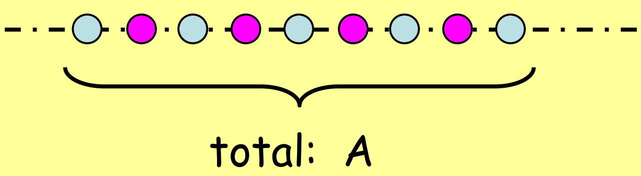
Binding energy of products is greater than the sum of the binding energies of the initial species.

Fission reactions at large *A* release energy because the products have greater binding energy per nucleon than the initial species:

$$^{235}\mathrm{U} + \mathrm{n} \Rightarrow \underbrace{{}^{100}\mathrm{X} + {}^{133}\mathrm{Y}}_{\text{distribution of final products}} + 3\mathrm{n} + 200\ \mathrm{MeV}$$

1. Volume and Surface terms:

First consider a 1-dimensional row of nucleons with interaction energy per pair = ε



$$B = \sum_{i=1}^{A} 2\varepsilon - \Delta = 2\varepsilon A - \Delta$$

each has 2 neighbors

correction for the ends

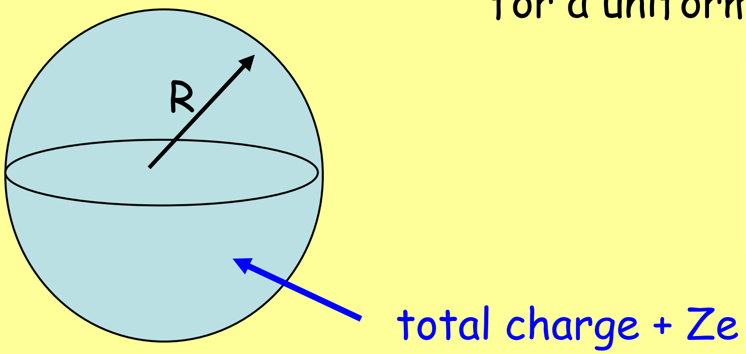
$$\Rightarrow \quad \frac{B}{A} = 2\varepsilon - \frac{\Delta}{A}$$

Approximately constant, with end effects relatively smaller at large *A*.

By analogy, for a 3-d nucleus, **there should be both volume and surface terms with the opposite sign**, the surface nucleons having less binding energy:

$$B = a_V A - a_S A^{2/3} \quad \Rightarrow \quad \frac{B}{A} = a_V - a_S A^{-1/3}$$

2. Coulomb term:



for a uniform sphere,

$$E_{Coul} = \int \frac{q(r)\, dq}{4\pi\varepsilon_o\, r} = \frac{3}{5}\, \frac{(Ze)^2}{4\pi\varepsilon_o R}$$

This effect increases the total energy and so decreases the binding energy.

Simple model: $\Delta B = -\, a_C\, Z^2\, A^{-1/3}$

But this is not quite right, because in a sense it includes the Coulomb self energy of a single proton by accounting for the integral from 0 to $r_p \sim 0.8$ fm. The nucleus has fuzzy edges anyway, so we will have to fit the coefficient a_c to mass data.

Solution: let ΔB scale as the number of proton pairs and include a term:

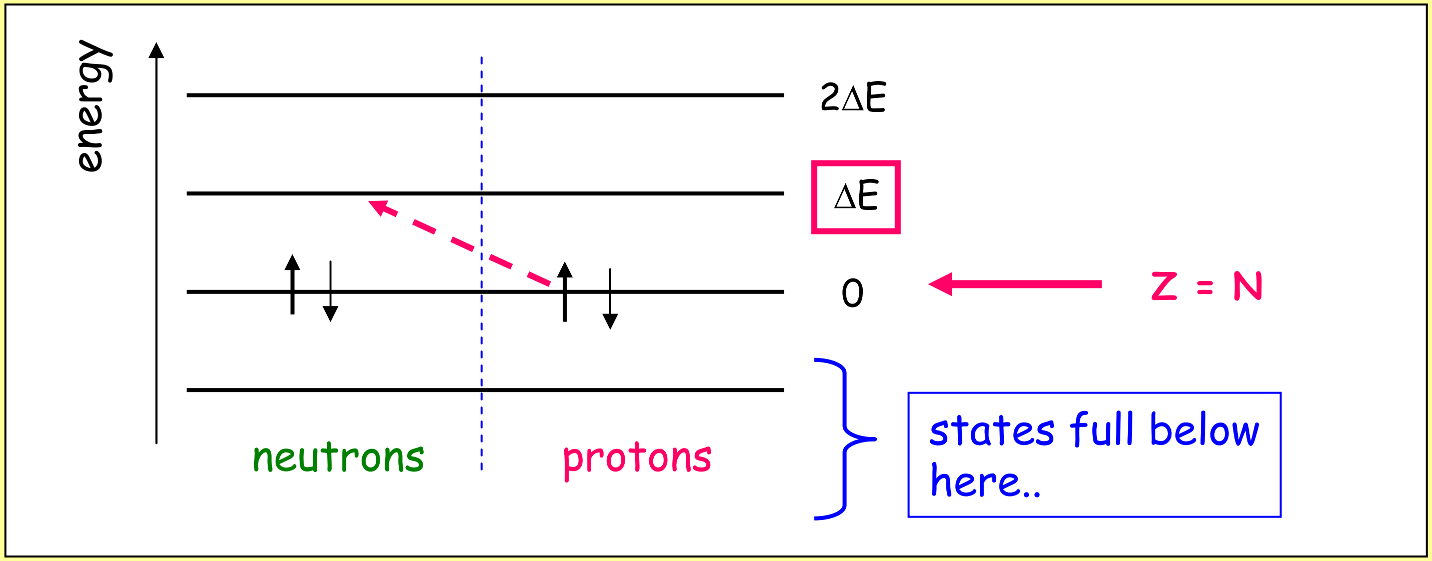
$$\Delta B = -\, a_C\, Z\,(Z-1)\, A^{-1/3} \;\Rightarrow\; \frac{\Delta B}{A} = -\, a_C\, Z\,(Z-1)\, A^{-4/3}$$

3. Symmetry Term:

So far, our formula doesn't account for the tendency for light nuclei to have Z = N. The nuclear binding energy ultimately results from filling allowed energy levels in a potential well V(r). **The most efficient way to fill these levels is with Z = N:**

Simplest model: identical nucleons as a Fermi gas, i.e. noninteracting spin- $\frac{1}{2}$ particles in a box. Two can occupy each energy level. The level spacing ~ 1/A.

A mismatch between Z and N costs an energy price of ΔE at fixed A as shown.



$$\Delta B = -a_A (Z-N)^2 A^{-1} = -a_A (A-2Z)^2 A^{-1}$$

4. Pairing Term:

Finally, recall from slide 1 that for the case of **even A**, there are 177 stable nuclei with Z and N both even, and only 6 with Z and N both odd. **Why?**

→ Configurations for which protons and neutrons separately can form pairs must be much more stable. All the even-even cases have $J^\pi = 0^+$, implying that neutrons and protons have lower energy when paired to total angular momentum zero.

Solution: add an empirical **pairing term** to the binding energy formula:

$$\Delta B_{pair} \equiv \delta = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix} a_p \, A^{-3/4}$$

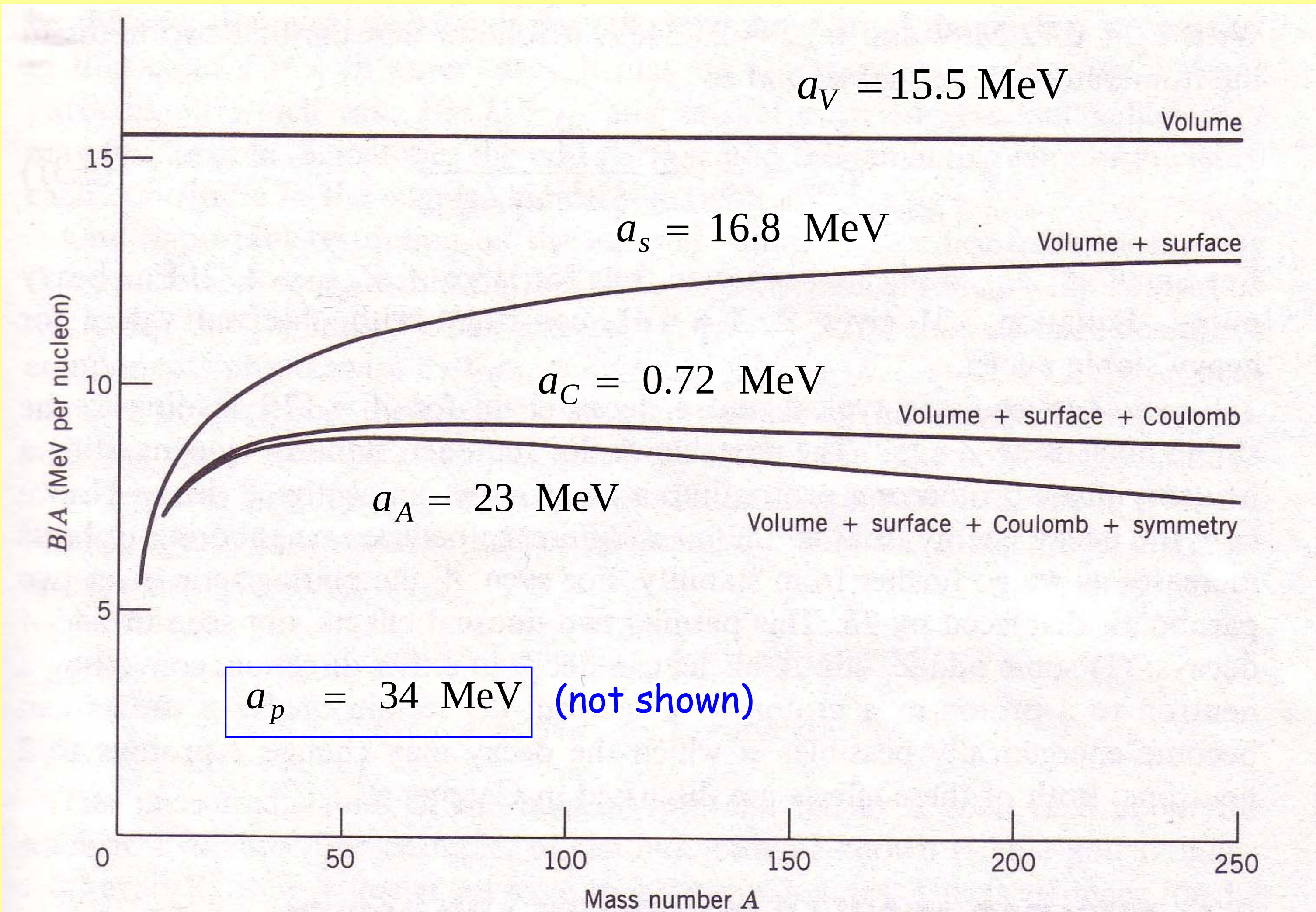
with +1 for even-even, 0 for even-odd, and -1 for odd-odd

Full expression:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C \, Z\,(Z-1)\, A^{-1/3} - a_A \,(A-2Z)^2 \, A^{-1} + \delta$$

Fitting of coefficients to data:

$$B(Z,A) = a_V A - a_S A^{2/3} - a_C\, Z\,(Z-1)\, A^{-1/3} - a_A\, (A-2Z)^2\, A^{-1} + \delta$$

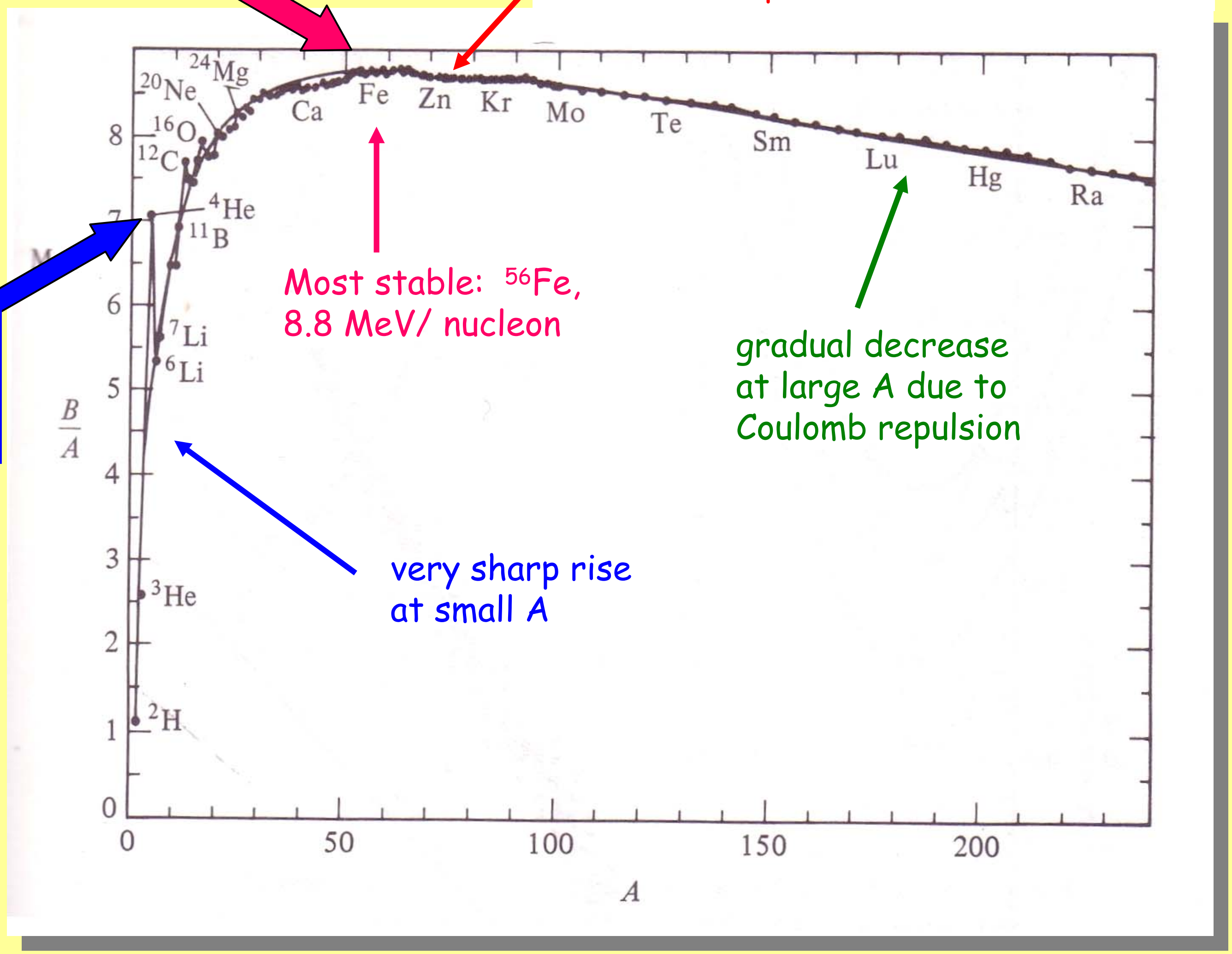


$a_p = 34$ MeV (not shown)

One more look at the Binding Energy per Nucleon curve:

Solid line: fit to the semi-empirical formula

almost flat, apart from Coulomb effects



some large oscillations at small mass

Most stable: ^{56}Fe, 8.8 MeV/ nucleon

gradual decrease at large A due to Coulomb repulsion

very sharp rise at small A